Long-Wave Elastic Anisotropy Produced by Horizontal Layering

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Abstract. A horizontally layered inhomogeneous medium, isotropic or transversely isotropic, is considered, whose properties are constant or nearly so when averaged over some vertical height l'. For waves longer than l' the medium is shown to behave like a homogeneous, or nearly homogeneous, transversely isotropic medium whose density is the average density and whose elastic coefficients are algebraic combinations of averages of algebraic combinations of the elastic coefficients of the original medium. The nearly homogeneous medium is said to be 'long-wave equivalent' to the original medium. Conditions on the five elastic coefficients of a homogeneous transversely isotropic medium are derived which are necessary and sufficient for the medium to be 'long-wave equivalent' to a horizontally layered isotropic medium. Further conditions are also derived which are necessary and sufficient for the homogeneous medium to be 'long-wave equivalent' to a horizontally layered isotropic medium consisting of only two different homogeneous isotropic materials. Except in singular cases, if the latter two-layered medium exists at all, its proportions and elastic coefficients are uniquely determined by the elastic coefficients of the homogeneous transversely isotropic medium. The observed variations in crustal *P*-wave velocity with depth, obtained from well logs, are shown to be large enough to explain some of the observed crustal anisotropies as due to layering of isotropic material.

1. INTRODUCTION

It is our purpose in the present paper to discuss the propagation of long seismic waves in a finely layered, horizontally stratified, transversely isotropic, elastic medium whose axis of symmetry is vertical. By 'long waves' and 'fine layering' we mean the following: we pick a length l' long enough so that the elastic properties of the medium vary appreciably over a length l'. We then consider only seismic waves in which the distance k^{-1}, over which the displacements change by an appreciable fraction of their values, is much larger than l'. We show that the variations in the medium which have vertical scales less than l' can be averaged out, so that the medium can be replaced by an equivalent but less wildly varying medium, at least in discussing waves for which $kl' \ll 1$.

In particular, we show that a horizontally stratified, continuously or discretely varying, isotropic medium whose Lamé parameters and density have position-independent averages over any vertical distance l' behaves like a homogeneous transversely isotropic (HTI) medium with a vertical symmetry axis whose elastic coefficients can be calculated explicitly as algebraic combinations of averages of algebraic combinations of the Lamé parameters of the original medium and whose density is the average density of the original medium.

The following question is also examined: Given the five elastic coefficients of a stable HTI medium, is there a layered isotropic medium which has these five elastic constants when it reacts, as a transversely isotropic medium, to long waves? The geophysical question at issue here is whether the apparent anisotropies in the earth's crust, observed by seismic prospecting with long waves [*Uhrig and Van Melle*, 1955], may be due to a fine horizontal layering of different isotropic rocks. The result of the theoretical investigation is that there are stable HTI materials which cannot be modeled by any stack of stable isotropic layers. Any stable HTI material which can be modeled for long waves by a stable, isotropic, layered medium can be modeled by a stack of isotropic stable, homogeneous (ISH) layers of just three different types. There are stacks of ISH layers of three types which cannot be modeled by any stack of two different types of ISH material. Finally, if a stable HTI material can be modeled for long waves by a stack of two ISH materials, then, except in singular cases, the proportions and properties of the two materials are uniquely determined by the transversely isotropic material they model.

This last result will be of some interest if it should develop that the second layer under the oceans consists of a relatively uniform isotropic sediment penetrated by occasional isotropic volcanic sills. The measurement of any five independent propagation velocities in the second layer would then determine the proportions and properties of the sills and the sediment to within an unknown scale factor for density.

The other application of the above results is to the propagation of long surface and body waves in crustal rock. It is concluded that some of the anisotropies observed in the crust can be explained by vertical variations in isotropic properties as large as those observed in well logs.

2. Summary of Previous Work

The problem of elastic wave propagation in finely layered media has been treated by a number of authors, all of whom except *Thomson* [1950], *Helbig* [1958], and *Anderson* [1961] have restricted themselves to what we shall call a periodic, isotropic, two-layered (PITL) medium: a medium periodic in the vertical direction and consisting of alternating isotropic layers of thicknesses h_1, h_2, having constant Lamé parameters λ_1, μ_1, and λ_2, μ_2, and constant densities ρ_1, ρ_2.

Riznichenko [1949] calculated, for long compression waves, the velocities of propagation in the vertical and horizontal directions, treating the medium as if it were locally static in order to get average stress-strain relations.

Thomson [1950] gave the formal solution for waves of arbitrary wavelength in a medium consisting of any number of different homogeneous isotropic layers; he found the displacements and vertical stresses at any interface by multiplying the surface displacements and stresses by a product of propagator matrices, one matrix for each layer between the interface and the surface. This technique has lent itself well to numerical calculation of dispersion relations [*Haskell*, 1953], but is rather cumbersome for our purposes. At any rate, no one has taken the limit of the matrix products for small wave number, a procedure which ought to yield the results of the present paper, but which we shall not use.

Postma [1955] gave explicit formulas for the five elastic coefficients of the HTI medium which is long-wave equivalent to any PITL medium. The results are very complicated and are not expressed as averages; therefore they do not suggest a generalization to nonperiodic media or media in which the Lamé parameters and density can take more than two values. Postma gave some inequalities which must be satisfied by the five elastic coefficients of a transversely isotropic medium if it is long-wave equivalent to a PITL medium. These inequalities are not exhaustive, however; their satisfaction by a transversely isotropic medium does not ensure that it can be modeled by a PITL medium, or, for that matter, by a stratified isotropic medium.

White and Angona [1955] calculated the horizontal and vertical propagation velocities of long compression and shear waves in a PITL medium, thus generalizing Riznichenko's result. Their paper is not equivalent to Postma's, as their conclusion (that the five velocities they calculated determine the five elastic constants of the long-wave equivalent HTI medium) is incorrect. The error was corrected by *Rytov* [1956].

Rytov completely and definitely solved the problem of the propagation of plane waves in a PITL medium. He used *Floquet*'s [1883] theorem, and hence his method is applicable to any medium whose properties vary periodically in the vertical direction. By examining the limiting case of small wave number k, he showed that the fractional error introduced by the other authors' long-wavelength approximation is of the order of $(kh)^2$, h being the vertical distance over which the properties of the medium are periodic. (The present author has seen only *Brekhovskikh*'s [1960] description of Rytov's work.)

Helbig [1958] expressed Postma's formulas as averages and generalized them to the multilayered case, but did not consider the possibility that the layers were intrinsically anisotropic. He gave some inequalities on Postma's elastic coefficients based on the hypothesis that the Lamé parameter λ be positive. This is not a stability condition, and Helbig did not try to justify it. His inequalities, like Postma's, were not exhaustive. (The author learned of Helbig's work from D. Anderson.)

Anderson [1961] has generalized Haskell's method to anisotropic layered media, but has not examined the long-wave limit.

Despite the completeness with which the problem of the PITL medium has been solved, the present author feels that at least two gaps remain in our knowledge of long waves in finely layered media, and he proposes to try to fill them in the following discussion. The first is the question of how to treat media containing layers of three or more kinds of rock when there is no vertical periodicity of properties (and, incidentally, when the separate layers may be intrinsically anisotropic). The other is the question of which HTI media can be modeled by stacking layers of isotropic media.

3. The Averaging Technique

Let x_1, x_2, x_3 be the position coordinates in a cartesian coordinate system. Let s_1, s_2, s_3 be the cartesian components of displacement of an elastic medium whose properties are independent of x_1 and x_2 but may vary with x_3. Let $w(x_3)$ be any continuous weighting function that averages over a length l'. That is, let $w(x_3)$ have these properties:

$$w(x_3) \geq 0 \qquad w(\pm\infty) = 0$$

$$\int_{-\infty}^{\infty} w(x_3)\, dx_3 = 1 \qquad \int_{-\infty}^{\infty} x_3 w(x_3)\, dx_3 = 0$$

$$\int_{-\infty}^{\infty} x^2 w(x_3)\, dx_3 = l'^2$$

Then, if $f(x_3)$ is any function,

$$\langle f\rangle(x_3) = \int_{-\infty}^{\infty} w(\zeta - x_3) f(\zeta)\, d\zeta \qquad (1)$$

is the average of f over a distance roughly l' around the position x_3. This is a moving average; that is, it depends on x_3. Effectively, $\langle f\rangle(x_3)$ is $f(x_3)$ with those wavelengths removed which are less than l'. The functional dependence of $\langle f\rangle(x_3)$ on x_3 will not usually be shown explicitly; the average will be written simply $\langle f\rangle$.

If f is a function of x_1, x_2 and x_3, then

$$\left\langle \frac{\partial f}{\partial x_i}\right\rangle = \frac{\partial}{\partial x_i}\langle f\rangle \qquad i = 1, 2, 3 \qquad (2)$$

The first two of these formulas are obvious, and the third follows on integrating (1) by parts for $\langle \partial f/\partial x_3\rangle$.

The only approximation that we make in the present paper is the following: if $f(x_3)$ is nearly constant when x_3 changes by no more than l', while $g(x_3)$ may vary by a large fraction over this distance, then, approximately,

$$\langle fg\rangle = f\langle g\rangle \qquad (3)$$

Now suppose that for each x_3 the medium is transversely isotropic with a vertical axis of symmetry. Then [*Stoneley*, 1949] the stress-strain relations in the medium can be written

$$\left.\begin{aligned}
T_{11} &= a\frac{\partial s_1}{\partial x_1} + b\frac{\partial s_2}{\partial x_2} + f\frac{\partial s_3}{\partial x_3}\\
T_{22} &= b\frac{\partial s_1}{\partial x_1} + a\frac{\partial s_2}{\partial x_2} + f\frac{\partial s_3}{\partial x_3}\\
T_{33} &= f\frac{\partial s_1}{\partial x_1} + f\frac{\partial s_2}{\partial x_2} + c\frac{\partial s_3}{\partial x_3}\\
T_{23} &= l\left(\frac{\partial s_2}{\partial x_3} + \frac{\partial s_3}{\partial x_2}\right)\\
T_{13} &= l\left(\frac{\partial s_1}{\partial x_3} + \frac{\partial s_3}{\partial x_1}\right)\\
T_{12} &= m\left(\frac{\partial s_1}{\partial x_2} + \frac{\partial s_2}{\partial x_1}\right)
\end{aligned}\right\} \qquad (4)$$

Of the six elastic parameters *a*, *b*, *c*, *f*, *l*, and *m*, only five are independent, since

$$a = b + 2m \qquad (5)$$

The elastic coefficient l should not be confused with the length l'.

Consider an infinite horizontal slab of vertical thickness $L' \gg l'$, consisting of discrete horizontal layers so thin that when averaged over a vertical distance l' all properties of the slab are nearly independent of x_3. All properties of the slab are assumed independent of x_1 and x_2. If the slab is subjected on its top and bottom to the same static stresses T_{13}, T_{23}, and T_{33}, independent of x_1 and x_2, then throughout the deformed slab T_{13}, T_{23}, and T_{33} will be constant. Furthermore, s_1, s_2, and s_3 will be continuous and will vary linearly in each layer with x_3 derivatives which vary widely from layer to layer: however, there will be constants N_1, N_2, and N_3, such that $|s_i(x_3) - N_i x_3| \ll L' |N_i|$. Thus T_{13}, T_{23}, T_{33}, $\partial s_i/\partial x_1$, and $\partial s_i/\partial x_2$ all vary very slowly or not at all in the slab. On the other hand, T_{11}, T_{12}, T_{22}, $\partial s_1/\partial x_3$, $\partial s_2/\partial x_3$, and $\partial s_3/\partial x_3$ all vary by large fractions from layer to layer in the slab because of the different elastic properties of the layers.

Since any continuously variable medium can be approximated arbitrarily closely by discretely layered media, in a continuously variable slab subjected to constant static surface stresses, T_{13}, T_{23}, and T_{33} will be constant, and the values of $\partial s_i/\partial x_1$ and $\partial s_i/\partial x_2$ will be smoothly varying functions of x_3 on which are superposed very small wiggles, while T_{11}, T_{12}, T_{22}, $\partial s_1/\partial x_3$, $\partial s_2/\partial x_3$, and $\partial s_3/\partial x_3$ will vary widely and rapidly with x_3.

Finally, if the stresses T_{13}, T_{23}, and T_{33} vary only slightly in a horizontal or vertical distance l', as in an elastic wave with wave number $k \ll l'^{-1}$, the above remarks remain approximately true. This observation amounts to stating that in an elastic wave of wave number k the stresses inside a piece of the medium whose diameter is much less than k^{-1} can be calculated from the stresses on the surface of the piece as if it were in static equilibrium. This point has already been made by *Riznichenko* [1949] and *Postma* [1955].

The manner in which the small-scale large-amplitude variations in T_{11}, T_{12}, T_{22}, $\partial s_1/\partial x_3$, $\partial s_2/\partial x_3$, and $\partial s_3/\partial x_3$, are produced by the small-scale large-amplitude variations in the properties of the medium can be exhibited explicitly by solving (4) for these six rapidly varying stress and displacement field variables. The result is

$$\left.\begin{aligned}
\frac{\partial s_1}{\partial x_3} &= \frac{1}{l}\,T_{13} - \frac{\partial s_3}{\partial x_1}\\
\frac{\partial s_2}{\partial x_3} &= \frac{1}{l}\,T_{23} - \frac{\partial s_2}{\partial x_3}\\
\frac{\partial s_3}{\partial x_3} &= \frac{1}{c}\,T_{33} - \frac{f}{c}\left(\frac{\partial s_1}{\partial x_1} + \frac{\partial s_2}{\partial x_2}\right)\\
T_{11} &= \left(a - \frac{f^2}{c}\right)\frac{\partial s_1}{\partial x_1} + \left(b - \frac{f^2}{c}\right)\frac{\partial s_2}{\partial x_2} + \frac{f}{c}\,T_{33}\\
T_{22} &= \left(b - \frac{f^2}{c}\right)\frac{\partial s_1}{\partial x_1} + \left(a - \frac{f^2}{c}\right)\frac{\partial s_2}{\partial x_2} + \frac{f}{c}\,T_{33}\\
T_{12} &= m\left(\frac{\partial s_1}{\partial x_2} + \frac{\partial s_2}{\partial x_1}\right)
\end{aligned}\right\} \quad (6)$$

All the field variables on the right in (6) vary slowly with x_3. The rapid variations with x_3 of the field variables on the left are produced by the rapid variations of the elastic coefficients.

The advantage of writing the stress-strain relations in the form of (6) is that these equations contain no products of a rapidly varying field variable and a rapidly varying elastic parameter. Thus when equations 6 are averaged over a vertical distance l' by means of the weighting function w, formula 3 can be applied to the averages on the right. Computing averages of derivatives by formula 2, we obtain from (6)

$$\left.\begin{aligned}
\frac{\partial\langle s_1\rangle}{\partial x_3} &= \left\langle\frac{1}{l}\right\rangle\langle T_{13}\rangle - \frac{\partial}{\partial x_1}\langle s_3\rangle\\
\frac{\partial\langle s_2\rangle}{\partial x_3} &= \left\langle\frac{1}{l}\right\rangle\langle T_{23}\rangle - \frac{\partial}{\partial x_2}\langle s_3\rangle\\
\frac{\partial\langle s_3\rangle}{\partial x_3} &= \left\langle\frac{1}{c}\right\rangle\langle T_{33}\rangle - \left\langle\frac{f}{c}\right\rangle\left(\frac{\partial}{\partial x_1}\langle s_1\rangle + \frac{\partial}{\partial x_2}\langle s_2\rangle\right)\\
\langle T_{11}\rangle &= \left\langle a - \frac{f^2}{c}\right\rangle\frac{\partial}{\partial x_1}\langle s_1\rangle + \left\langle b - \frac{f^2}{c}\right\rangle\frac{\partial}{\partial x_2}\langle s_2\rangle + \left\langle\frac{f}{c}\right\rangle\langle T_{33}\rangle\\
\langle T_{22}\rangle &= \left\langle b - \frac{f^2}{c}\right\rangle\frac{\partial}{\partial x_1}\langle s_1\rangle + \left\langle a - \frac{f^2}{c}\right\rangle\frac{\partial}{\partial x_2}\langle s_2\rangle + \left\langle\frac{f}{c}\right\rangle\langle T_{33}\rangle\\
\langle T_{12}\rangle &= \langle m\rangle\left(\frac{\partial}{\partial x_2}\langle s_1\rangle + \frac{\partial}{\partial x_1}\langle s_2\rangle\right)
\end{aligned}\right\} \quad (7)$$

If equations 7 are solved for the averaged stresses, we obtain relations between averaged stresses and the strains calculated from averaged displacements:

$$\left.\begin{aligned}
\langle T_{11}\rangle &= A\,\frac{\partial}{\partial x_1}\langle s_1\rangle + B\,\frac{\partial}{\partial x_2}\langle s_2\rangle + F\,\frac{\partial}{\partial x_3}\langle s_3\rangle\\
\langle T_{22}\rangle &= B\,\frac{\partial}{\partial x_1}\langle s_1\rangle + A\,\frac{\partial}{\partial x_2}\langle s_2\rangle + F\,\frac{\partial}{\partial x_3}\langle s_3\rangle\\
\langle T_{33}\rangle &= F\,\frac{\partial}{\partial x_1}\langle s_1\rangle + F\,\frac{\partial}{\partial x_2}\langle s_2\rangle + C\,\frac{\partial}{\partial x_2}\langle s_3\rangle\\
\langle T_{23}\rangle &= L\left(\frac{\partial}{\partial x_3}\langle s_2\rangle + \frac{\partial}{\partial x_2}\langle s_3\rangle\right)\\
\langle T_{13}\rangle &= L\left(\frac{\partial}{\partial x_3}\langle s_1\rangle + \frac{\partial}{\partial x_1}\langle s_3\rangle\right)\\
\langle T_{12}\rangle &= M\left(\frac{\partial}{\partial x_2}\langle s_1\rangle + \frac{\partial}{\partial x_1}\langle s_2\rangle\right)
\end{aligned}\right\} \quad (8)$$

The effective elastic coefficients in equations 8 are

$$\left.\begin{aligned} A &= \langle a - f^2c^{-1}\rangle + \langle c^{-1}\rangle^{-1}\langle fc^{-1}\rangle^2 \\ B &= \langle b - f^2c^{-1}\rangle + \langle c^{-1}\rangle^{-1}\langle fc^{-1}\rangle^2 \\ C &= \langle c^{-1}\rangle^{-1} \\ F &= \langle c^{-1}\rangle^{-1}\langle fc^{-1}\rangle \\ L &= \langle l^{-1}\rangle^{-1} \\ M &= \langle m\rangle \end{aligned}\right\} \quad (9)$$

A number of remarks are in order about these relations (9) between average stresses and average strains. First, note that $a = b + 2m$ and therefore $A = B + 2M$. Consequently, the averaged stress-strain relations (8) are those of an elastic transversely isotropic solid with vertical axis of symmetry. This new solid will be said to be 'long-wave equivalent' to the original, more strongly layered solid. The elastic constants of the new, more nearly homogeneous solid which is long-wave equivalent to the old, more strongly inhomogeneous solid are not simply averages of the corresponding elastic constants in the more inhomogeneous solid, except that $M = \langle m\rangle$. Also note that if a, b, c, f, l, and m are all constant, then, as expected, $A = a$, $B = b$, $C = c$, $F = f$, $L = l$, and $M = m$; the new more nearly homogeneous solid is identical with the original one.

In case $a = a_0 + \delta a$, $b = b_0 + \delta b$, etc., where $|\delta a/a_0| \ll 1$, $|\delta b/b_0| \ll 1$, etc., $\langle \delta a\rangle = \langle \delta b\rangle = \cdots = 0$, and a_0, b_0, $\cdots$ are constant, expressions 9 can be simplified by neglecting all but the lowest-order terms in δa, δb, $\cdots$. The result is

$$A = a_0 - \frac{f_0^2}{c_0}\left\langle\left(\frac{\delta f}{f_0}\right)^2\right\rangle$$

$$B = b_0 - \frac{f_0^2}{c_0}\left\langle\left(\frac{\delta f}{f_0}\right)^2\right\rangle$$

$$C = c_0 - c_0\left\langle\left(\frac{\delta c}{c_0}\right)^2\right\rangle$$

$$F = f_0 - f_0\left\langle\frac{\delta f}{f_0}\,\frac{\delta c}{c_0}\right\rangle$$

$$L = l_0 - l_0\left\langle\left(\frac{\delta l}{l_0}\right)^2\right\rangle$$

$$M = m_0$$

In using equations 8 it should be recalled that the *values* (but not necessarily the derivatives) of T_{13}, T_{23}, T_{33}, s_1, s_2, s_3, $\partial s_i/\partial x_1$, and $\partial s_i/\partial x_2$ are approximately the same as their averages:

$$T_{i3} = \langle T_{i3}\rangle \qquad s_i = \langle s_i\rangle$$

$$\frac{\partial s_i}{\partial x_1} = \frac{\partial}{\partial x_1}\langle s_i\rangle \qquad \frac{\partial s_i}{\partial x_2} = \frac{\partial}{\partial x_2}\langle s_i\rangle$$

We still must examine the equations of motion of the layered medium. Neglecting gravity, we obtain

$$\rho\,\frac{\partial^2 s_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j} \qquad (10)$$

If we average both sides of these equations over the vertical distance l', using the weighting function w, we obtain

$$\langle\rho\rangle\,\frac{\partial^2}{\partial t^2}\langle s_i\rangle = \frac{\partial}{\partial x_j}\langle T_{ij}\rangle \qquad (11)$$

Here formulas 2 and 3 have been applied. It thus develops that for waves much longer than l' the equations of motion (11) and the stress-strain relations (8) for the average stresses and displacements are precisely those of a transversely isotropic medium with vertical axis of symmetry, whose elastic parameters (9) are smoother than those of the original medium, all variations on vertical scales of l' or less having been removed from the elastic parameters.

To find how the real medium moves in the presence of waves with wavelengths much longer than l', we solve (9) and (11) for the equivalent smoothed medium, thus obtaining the averaged stresses $\langle T_{ij}\rangle$ and displacements $\langle s_i\rangle$. The actual stresses T_{13}, T_{23}, and T_{33} and the actual displacements are the same as their averages, to within our accuracy, while the values of T_{11}, T_{12}, T_{22}, $\partial s_1/\partial x_3$, $\partial s_2/\partial x_3$, and $\partial s_3/\partial x_3$ can be found from (6), the field variables on the right being replaced by their averages.

As *Stoneley* [1949] has shown, in a HTI medium the square of the velocity of vertical propagation is c/ρ for compression waves and l/ρ for shear waves. The velocity of horizontal propagation is a/ρ for compression waves, l/ρ for SV waves, and m/ρ for SH waves. From the above remarks, we conclude that, in a layered transversely isotropic medium in which A, B, C, F, L, M, and $\langle\rho\rangle$ are constant, the corresponding velocities for waves much longer than l' should be $C/\langle\rho\rangle$

for vertical P velocity, $L/\langle\rho\rangle$ for vertical S velocity, $A/\langle\rho\rangle$ for horizontal P velocity, $L/\langle\rho\rangle$ for horizontal SV, and $M/\langle\rho\rangle$ for horizontal SH. The elastic constant F does not appear among these velocities, a point overlooked by White and Angona, which prevents the use of these velocities in determining all the elastic constants of the medium.

The long-wavelength part of the impulse response of a layered medium having constant average properties ought to be the long-wavelength part of *Kraut*'s [1962] calculated impulse response of a HTI medium, the appropriate elastic coefficients being given by (9). The limitation to long waves has the following effect. At a surface detector, the first arrival travels with speed $(a/\rho)^{1/2}$ in most real HTI media. In a layered isotropic medium, $(A/\langle\rho\rangle)^{1/2}$ is slower than the speeds of compression waves in some of the layers, so that the first arrival is earlier than the above theory would indicate. However, if the receiver is a low-pass filter which 'sees' only waves longer than l', the head waves carried by the fast layers will presumably die out rapidly with distance from the source (we have not examined this question), and the first large arrival will come in with velocity $(A/\langle\rho\rangle)^{1/2}$ if the separation of source and receiver is several times l'.

Comparison of (9) and (11) indicates that in the averaging process which converts a finely layered, highly variable medium to a smoothed, transversely isotropic, long-wave equivalent (STILWE) medium, the averaging which occurs in the equations of motion is quite simple, while the averaging in the stress-strain relations is not. The remainder of the present paper is devoted to an algebraic discussion of the stress-strain averages when the highly variable real medium is locally isotropic. The goal of this discussion is to find how far apparent anisotropy in the earth's crust can be due to a layering of isotropic media.

4. Locally Isotropic Layered Media

In an isotropic medium, equations 4 become the Lamé relations. If λ and μ are the Lamé parameters,

$$a = c = \lambda + 2\mu \qquad b = f = \lambda \qquad l = m = \mu \tag{12}$$

These expressions can be substituted into (9) to obtain the elastic coefficients of the STILWE medium. The result, which can be shown to agree with *Postma*'s [1955] result when the real medium is periodic and two-layered, is

$$\left.\begin{aligned}
A &= \left\langle\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\right\rangle + \left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1}\left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle^{2}\\
B &= \left\langle\frac{2\lambda\mu}{\lambda+2\mu}\right\rangle + \left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1}\left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle^{2}\\
C &= \left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1}\\
F &= \left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1}\left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle\\
L &= \left\langle\frac{1}{\mu}\right\rangle^{-1}\\
M &= \langle\mu\rangle
\end{aligned}\right\} \tag{13}$$

These expressions are algebraic combinations of averages of algebraic combinations of λ and μ. Most of the algebra can be eliminated from the relation between the elastic parameters of the real isotropic medium and the STILWE medium by defining new elastic parameters in the two media. In the real isotropic medium we shall use the elastic parameters μ and θ, where

$$\theta = \frac{\mu}{\lambda + 2\mu} = 1 - \frac{1}{2(1-\sigma)} \tag{14}$$

σ being Poisson's ratio. The dimensionless parameter θ is the square of the ratio of shear velocity to compressional velocity. When $\sigma = \frac{1}{4}$, a common value among real materials, $\theta = \frac{1}{3}$. The range of θ and μ for which the medium is stable is

$$0 \le \theta \le \tfrac{3}{4} \qquad 0 \le \mu < \infty \tag{15}$$

In the STILWE medium we shall use the elastic parameters L, M, R, S, and T, where

$$\left.\begin{aligned}
R &= C^{-1}\\
S &= (4C)^{-1}(F^2 + 2MC - BC)\\
T &= (2C)^{-1}(C - F)
\end{aligned}\right\} \tag{16}$$

The elastic coefficients A, B, C, F, L, and M

are given in terms of L, M, R, S, and T by the formulas inverse to (16)

$$\left.\begin{aligned} A &= B + 2M \\ B &= 2M - 4S + R^{-1}(1 - 2T)^2 \\ C &= R^{-1} \\ F &= R^{-1}(1 - 2T) \end{aligned}\right\} \qquad (17)$$

In terms of the new elastic parameters the relations (9) between the real isotropic medium and the long-wave equivalent, transversely isotropic medium are simply

$$\left.\begin{aligned} L &= \langle 1/\mu \rangle^{-1} \\ M &= \langle \mu \rangle \\ R &= \langle \theta/\mu \rangle \\ S &= \langle \theta\mu \rangle \\ T &= \langle \theta \rangle \end{aligned}\right\} \qquad (18)$$

The question we propose to consider is this: Given elastic coefficients L, M, R, S, and T for a stable HTI medi m, is there a stable, isotropic, horizontally layered medium which is long-wave equivalent to the homogeneous medium? That is, are there functions $\mu(x_3)$ and $\theta(x_3)$, satisfying the stability conditions (15) or more stringent restrictions, which are related by (18) to the elastic coefficients of the given HTI medium?

5. Conditions for Stability and Isotropy

To discuss the question just raised, we must know the conditions on the elastic coefficients which ensure stability (that is, that no deformation have negative internal energy). The stability conditions (15) for an isotropic medium are well known. A necessary and sufficient condition for the stability of the transversely isotropic medium whose stress-strain relation is (4) is that the following matrix be positive semidefinite:

$$\begin{bmatrix} a & b & f & 0 & 0 & 0 \\ b & a & f & 0 & 0 & 0 \\ f & f & c & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \qquad (19)$$

The matrix (19) with $a = b + 2m$ is positive semidefinite if and only if all its principal minors are non-negative. It is sufficient for positive definiteness that all the upper left principal minors be positive. That this result cannot be extended to positive semidefiniteness is shown by the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

The resulting conditions are

$$l \geq 0 \qquad m \geq 0 \qquad b + m \geq 0$$
$$c \geq 0 \qquad (b + m)c \geq f^2 \qquad (20)$$

There are some further inequalities which, being consequences of (20), can be omitted.

Parenthetically we remark that if $b(x_3)$, $c(x_3)$, $f(x_3)$, $l(x_3)$, and $m(x_3)$ satisfy all of conditions 20 and if $a(x_3) = b(x_3) + 2m(x_3)$, then the coefficients A, B, C, F, L, and M defined by (9) also satisfy all of conditions 20. The proof is straightforward and will be omitted. It follows that if the real medium is stable, so is the fictitious STILWE medium. *Postma* [1955] constructed a PITL medium, both of whose layers are stable and whose STILWE medium violates one of the inequalities which Postma quoted *Rudski* [1911] as saying *must* be satisfied by the elastic coefficients of a transversely isotropic medium. In fact, Rudski said only that his inequalities are *usually* satisfied by real anisotropic materials. His inequalities include, besides the stability conditions (20), the further inequalities $a \geq l$, $c \geq l$, and $c \geq m$; it is the third of these latter inequalities which Postma's example violates.

For the STILWE medium it will be convenient to have the stability conditions (20) on B, C, F, L, and M rewritten in terms of L, M, R, S, and T. The result is

$$L \geq 0 \quad M \geq 0 \quad R \geq 0 \quad \tfrac{3}{4}M \geq S \quad (21)$$

Having noted the stability conditions for a transversely isotropic medium, we now seek conditions on the elastic coefficients that are necessary and sufficient for isotropy of the medium. From (12) these conditions are clearly

$$b = f \qquad l = m \qquad a = c = b + 2m \quad (22)$$

A smoothed equivalent medium will appear to be isotropic for long waves, even though it is layered, if its effective elastic coefficients A, B, C, F, L, and M satisfy (22). In terms of the elastic coefficients L, M, R, S, and T, these conditions for isotropy are

$$L = M \qquad S = MT \qquad T = MR \tag{23}$$

For reasons which will appear later, we prefer to write these equations in the following equivalent but more complicated form:

$$\left.\begin{aligned} L &= M \\ T^2 &= RS \\ (\tfrac{3}{4} - T)^2 &= (\tfrac{3}{4}L^{-1} - R)(\tfrac{3}{4}M - S) \end{aligned}\right\} \tag{24}$$

Our task now is to see which stable HTI media are long-wave equivalent to some stable, layered, isotropic medium, or to a particular kind of such medium.

6. The Case of Constant Rigidity

We ask first what conditions beside the stability conditions (21) are necessary and sufficient to insure that L, M, R, S, and T are the elastic coefficients of a HTI medium which is long-wave equivalent to a stable, isotropic, layered medium with constant rigidity?

On the one hand, if L, M, R, S, and T are the averaged coefficients of an isotropic layered medium with constant μ, we have, from (18), $L = M = \langle\mu\rangle = \mu$, $T = \langle\theta\rangle$, $R = M^{-1}T$, and $S = MT$. But these are exactly the conditions (23) ensuring that the long-wave equivalent, homogeneous medium be isotropic.

Conversely, if the homogeneous medium is stable and isotropic, it is equivalent to a stable isotropic medium with constant μ and θ, namely $\mu = M$, $\theta = T$.

Parenthetically we remark that inequalities 28, 29, and 30 show that if any one of the three isotropy conditions (24) is satisfied by the smoothed medium, and if the original layered isotropic medium has $0 < \theta < \frac{3}{4}$, then that isotropic medium must have constant μ.

To summarize, if a layered isotropic medium has constant μ, the STILWE medium is isotropic. This much was proved by *Postma* [1955] for periodic two-layered media. Conversely, if a transversely isotropic medium is isotropic, it can be the smoothed medium equivalent to an isotropic layered medium with constant μ, but it cannot be the smoothed medium equivalent to any layered isotropic medium with variable μ.

7. The Case of Constant Poisson Ratio

Somewhat more interesting physically is the following question: When is a stable HTI medium long-wave equivalent to a finely layered isotropic medium with constant θ and variable μ?

If L, M, R, S, and T are the elastic coefficients of a homogeneous medium which is long-wave equivalent to a stable, layered, isotropic medium with constant θ, then, by equations 18,

$$RL = T \qquad S = TM$$
$$0 \le T \le \tfrac{3}{4} \qquad L \le M \tag{25}$$

The first three of equations 25 are trivial consequences of (15) and (18). The fourth, $L \le M$, is proved as follows:

$$1 = \langle 1\rangle^2 = \langle\mu^{-1/2}\mu^{1/2}\rangle^2 \le \langle\mu^{-1}\rangle\langle\mu\rangle = L^{-1}M$$

The inequality in the above chain is Schwarz's inequality.

Conversely, we assert that if L, M, R, S, and T satisfy (21) and (25) they are the elastic coefficients of a HTI medium which is long-wave equivalent to a stable, layered, isotropic medium with constant θ. Obviously we must take $\theta = T$. Suppose we can find a $\mu(x_3)$ such that $\langle\mu^{-1}\rangle = L^{-1}$ and $\langle\mu\rangle = M$. Then, by (25) and the constancy of θ, $R = \langle\theta/\mu\rangle$ and $S = \langle\theta\mu\rangle$. The existence of the required $\mu(x_3)$ is shown by

Lemma 1. There is a function $\mu \ge 0$ such that

$$\langle\mu^{-1}\rangle = L^{-1} \qquad \langle\mu\rangle = M$$

if and only if $0 \le L \le \infty$, $0 \le M \le \infty$, and $L \le M$.

Proof. That $L = \langle\mu^{-1}\rangle^{-1}$ and $M = \langle\mu\rangle$ imply $L \le M$ was proved from Schwarz's inequality in proving (25). It remains to prove the converse. If L or M is 0 or ∞, or if $L = M$, the problem is trivial, and so we assume $0 < L < \infty$, $0 < M < \infty$, and $L < M$. We consider a layered isotropic medium of which a fraction p_1, distributed no matter how, has constant rigidity $u_1 > 0$ while the remaining fraction p_2 has constant rigidity $\mu_2 > \mu_1$. We hope to find p_1, p_2, μ_1, and μ_2 such that $p_1 + p_2 = 1$, $p_1\mu_1 + p_2\mu_2 = M$, and $p_1\mu_1^{-1} + p_2\mu_2^{-1} = L^{-1}$. Assume for the moment that μ_1 and μ_2 are known, and solve the first two equations for p_1 and p_2. The result is

$$p_1 = \frac{\mu_2 - M}{\mu_2 - \mu_1} \qquad p_2 = \frac{M - \mu_1}{\mu_2 - \mu_1}$$

To satisfy the conditions of the problem we must have $p_1 \geq 0$ and $p_2 \geq 0$; therefore $0 < \mu_1 \leq M \leq \mu_2$. Also, $p_1\mu_1^{-1} + p_2\mu_2^{-1} = L^{-1}$; therefore $u_1 + \mu_2 - M = L^{-1} \mu_1\mu_2$. We try $\mu_1 = M - \chi$ and $\mu_2 = M + \chi$. We must then have $0 < \chi < M$ and $M = L^{-1}(M^2 - \chi^2)$, or $\chi^2 = M(M - L) = M^2 - ML$. Thus $L < M$ is precisely the condition needed to ensure that χ is real and $0 < \chi < M$. This concludes the proof of lemma 1.

To summarize, the above arguments show that if a HTI medium is long-wave equivalent to a layered isotropic medium with constant θ, the elastic coefficients of the former medium satisfy conditions 21 and 25, and conversely. As can be seen from section 6, it is not true in general that the homogeneous medium which is equivalent to a layered isotropic medium with variable θ must fail to satisfy (25).

8. The General Case

Now we ask what are conditions on the elastic coefficients L, M, R, S, and T necessary and sufficient in order that there exist functions $\mu(x_3)$ (not constant) and $\theta(x_3)$ satisfying the strict isotropic stability conditions

$$0 < \mu < \infty \qquad 0 < \theta < \tfrac{3}{4} \tag{26}$$

almost everywhere and giving L, M, R, S, and T via (18). For simplicity, we do not consider the weak stability conditions (15). Our conclusion will be

Theorem 1. Such functions $\mu(x_3)$ and $\theta(x_3)$ exist if and only if

$$\begin{aligned} &0 < R < \tfrac{3}{4}L^{-1} \qquad 0 < S < \tfrac{3}{4}M \\ &T^2 < RS \qquad\qquad\quad 0 < T < \tfrac{3}{4} \\ &(\tfrac{3}{4} - T)^2 < (\tfrac{3}{4}L^{-1} - R)(\tfrac{3}{4}M - S) \end{aligned} \tag{27}$$

The proof is somewhat involved. First we need

Lemma 2. Suppose $f(x_3) \geq 0$ and $\phi(x_3) \geq 0$. Then $\langle\phi(f - 1)^2\rangle \leq \langle\phi(f - f^{-1})^2\rangle$.

Proof. Let $f = 1 + g$. Then $\langle\phi(f - f^{-1})^2\rangle = \langle\phi g^2(1 + f)^2/f^2\rangle \geq \langle\phi g^2\rangle = \langle\phi(f - 1)^2\rangle$. We also need

Lemma 3. There is a nonconstant function $\mu(x_3)$ such that $0 < \mu < \infty, \langle\mu\rangle < \infty, \langle\mu^{-1}\rangle < \infty$, $\langle\mu^{-1}\rangle = L^{-1}$, and $\langle\mu\rangle = M$ if and only if $0 < L < \infty, 0 < M < \infty$, and $L < M$.

Proof. The proof that if $0 < L < \infty$, $0 < M < \infty$, and $L < M$ then a function of the required kind exists has already been given in proving lemma 1. To prove the converse, note that if $M = 0$ then $\mu = 0$ almost everywhere, since μ is non-negative. But then $0 < M < \infty$. If $L = \infty$ then $\langle\mu^{-1}\rangle = 0$, so that $\mu^{-1} = 0$ almost everywhere, and $\langle\mu\rangle = \infty$. Thus, $0 < L < \infty$. Finally, as may be verified by direct calculation, $\langle(u^{1/2} M^{-1/2} - \mu^{-1/2} M^{1/2})^2\rangle = (M/L) - 1$, so that by lemma 2 with $\phi = 1$, $\langle(\mu^{1/2} M^{-1/2} - 1)^2\rangle \leq (M/L) - 1$. Thus $L \leq M$. But, by hypothesis, μ is not constant. Hence $L < M$.

Having established lemmas 2 and 3, we prove the first half of theorem 1. That is, if μ and θ exist, inequalities (27) follow. The inequalities $0 < R < 3L^{-1}/4, 0 < S < 3M/4$, and $0 < T < \tfrac{3}{4}$ are obvious from (18) and (26). To prove $T^2 < RS$, we note that

$$\langle(\mu^{1/2}T^{1/2}S^{-1/2} - \mu^{-1/2}T^{-1/2}S^{1/2})^2\theta\rangle = (RS - T^2)/T \tag{28}$$

Hence, by lemma 2,

$$\langle\theta(\mu^{1/2}T^{1/2}S^{-1/2} - 1)^2\rangle \leq (RS - T^2)/T \tag{29}$$

Thus $RS \geq T^2$, and, if $RS = T^2$,

$$\theta(\mu^{1/2}T^{1/2}S^{-1/2} - 1)^2 = 0$$

almost everywhere, contrary to the hypotheses that $0 < \theta$ and that μ not be constant. Therefore, $T^2 < RS$. The inequality $(\tfrac{3}{4} - T)^2 < (\tfrac{3}{4} L^{-1} - R)(\tfrac{3}{4} M - S)$ is proved in similar fashion by observing that

$$\begin{aligned} \langle(\tfrac{3}{4} - \theta)[\mu^{1/2}(\tfrac{3}{4}L^{-1} - R)^{1/2}(\tfrac{3}{4} - T)^{-1/2} \\ - \mu^{-1/2}(\tfrac{3}{4}L^{-1} - R)^{-1/2}(\tfrac{3}{4} - T)^{1/2}]^2\rangle \\ = [(\tfrac{3}{4}L^{-1} - R)(\tfrac{3}{4}M - S) \\ - (\tfrac{3}{4} - T)^2]/(\tfrac{3}{4} - T) \end{aligned}$$

whence, by lemma 2,

$$\begin{aligned} \langle(\tfrac{3}{4} - \theta)[\mu^{1/2}(\tfrac{3}{4}L^{-1} - R)^{1/2}(\tfrac{3}{4} - T)^{-1/2} - 1]^2\rangle \\ \leq [(\tfrac{3}{4}L^{-1} - R)(\tfrac{3}{4}M - S) \\ - (\tfrac{3}{4} - T)^2]/(\tfrac{3}{4} - T) \end{aligned} \tag{30}$$

Finally, we must prove the second half of theorem 1, that the inequalities (27) ensure the existence of functions μ (not constant) and θ satisfying conditions (26), and related to L, M, R, S, and T by (18). We consider an isotropic layered medium consisting of a fraction p_1 in which θ takes the constant value θ_1, and a fraction p_2 in which θ is the larger constant θ_2; $\theta_2 > \theta_1$. We denote the average of μ in the first fraction by $\langle\mu\rangle_1$ and its average in the second fraction by $\langle\mu\rangle_2$. We then hope to find p_1, p_2, θ_1, θ_2, and $\mu(x_3)$ such that

$$p_1 + p_2 = 1 \qquad p_1\theta_1 + p_2\theta_2 = T$$

$$p_1\langle\mu^{-1}\rangle_1 + p_2\langle\mu^{-1}\rangle_2 = L^{-1}$$

$$p_1\theta_1\langle\mu^{-1}\rangle_1 + p_2\theta_2\langle\mu^{-1}\rangle_2 = R$$

$$p_1\langle\mu\rangle_1 + p_2\langle\mu\rangle_2 = M$$

$$p_1\theta_1\langle\mu\rangle_1 + p_2\theta_2\langle\mu\rangle_2 = S$$

Solving the first two of these equations for p_1 and p_2, we have

$$p_1 = \frac{\theta_2 - T}{\theta_2 - \theta_1} \qquad p_2 = \frac{T - \theta_1}{\theta_2 - \theta_1}$$

The last four equations can then be solved for $\langle\mu\rangle_1$, $\langle\mu^{-1}\rangle$, $\langle\mu\rangle_2$, and $\langle\mu^{-1}\rangle_2$ in terms of θ_1, θ_2, L, M, R, S, and T:

$$\langle\mu^{-1}\rangle_1 = \frac{\theta_2 L^{-1} - R}{\theta_2 - T} \qquad \langle\mu^{-1}\rangle_2 = \frac{R - L^{-1}\theta_1}{T - \theta_1}$$

$$\langle\mu\rangle_1 = \frac{\theta_2 M - S}{\theta_2 - T} \qquad \langle\mu\rangle_2 = \frac{S - \theta_1 M}{T - \theta_1}$$

By lemma 3 we can find a positive function μ which satisfies the four equations given above if and only if

$$(\theta_2 L^{-1} - R)(\theta_2 M - S) > (\theta_2 - T)^2 \tag{31}$$

and

$$(R - \theta_1 L^{-1})(S - \theta_1 M) > (T - \theta_1)^2$$

The problem is solved if we can find a θ_1 and a θ_2 satisfying the above inequalities (31) and such that $0 < \theta_1 < T < \theta_2 < \frac{3}{4}$. Inequalities 27 and $0 < T < \frac{3}{4}$ ensure that we can do so by taking θ_1 very close to but greater than zero and θ_2 very close to but less than $\frac{3}{4}$. Theorem 1 is thus proved.

A number of consequences of theorem 1 are worth noting. First, and most important, inequalities 27 are much more restrictive than the stability conditions (21). That is, there are many stable HTI media which are not long-wave equivalent to any layered, stable, isotropic medium. If the elastic coefficients of an apparently transversely isotropic medium observed by long waves in the field fail to satisfy inequalities 27, then it is certain that some intrinsic anisotropy is present; no layered isotropic medium can reproduce the observations. On the other hand, if the field observations do satisfy all of inequalities 27, in principle it is possible that the observed material may be a finely layered isotropic material. The only way to confirm or eliminate this possibility is to look for a wavelength dependence of L, M, R, S, and T at shorter wavelengths, or to obtain an actual sample of the material.

A purely algebraic consequence of theorem 1 is this: From lemma 3, in a homogeneous material which is long-wave equivalent to an isotropic layered material of variable rigidity, $L < M$. But inequalities 27 imply that L, M, R, S, and T are the elastic constants of such a homogeneous material. Hence inequalities 27 must imply that $L < M$. This can be shown directly, but it involves some intricate algebra, which will be omitted.

Finally, on examining the proofs of lemma 1 and theorem 1 we see that we have proved more than is stated in theorem 1: A stable HTI medium whose elastic constants satisfy inequalities 27 is long-wave equivalent to a stable, layered, isotropic medium consisting of only four different kinds of material. Such a medium will be called four-layered, meaning not that four layers are present but that layers of four materials are present. It can be shown, by an intricate argument which will not be reproduced here, that in fact inequalities 27 imply that L, M, R, S, and T can be reproduced by a three-layered, stable, isotropic medium. Therefore, any stable HTI medium which is long-wave equivalent to a stable, layered, isotropic medium is long-wave equivalent to a stable, three-layered, isotropic medium. This, however, is the best we can do. As we shall see in the next section, two-layered isotropic materials are essentially less general.

9. The Two-Layered Case

In this case, the only one considered so far in the literature, we seek necessary and sufficient conditions that a stable HTI medium with elastic constants L, M, R, S, and T be long-wave equivalent to a stable, isotropic, horizontally layered medium containing only two different kinds of homogeneous isotropic material. A fraction p_1 of the medium has $\theta = \theta_1$ and $\mu = \mu_1$, and a fraction p_2 has $\theta = \theta_2$ and $\mu = \mu_2$. The constants θ_1, θ_2, μ_1, and μ_2 satisfy the stability inequalities (26). We must then solve the equations

$$p_1 + p_2 = 1 \qquad p_1\theta_1 + p_2\theta_2 = T$$
$$p_1\mu_1 + p_2\mu_2 = M \quad p_1\theta_1\mu_1 + p_2\theta_2\mu_2 = S$$
$$p_1\mu_1^{-1} + p_2\mu_2^{-1} = L^{-1} \tag{32}$$
$$p_1\theta_1\mu_1^{-1} + p_2\theta_2\mu_2^{-1} = R$$

These are six equations in six unknowns; hence we expect in general to be able to calculate p_1, p_2, θ_1, θ_2, μ_1, and μ_2 from field measurements of L, M, R, S, and T.

If any one of the equations $L = M$, $RS = T^2$, or $(\frac{3}{4}L^{-1} - R)(\frac{3}{4}M - S) = (\frac{3}{4} - T)^2$ is satisfied, we know from theorem 1 that equations 32 can have no solution with $\mu_1 \neq \mu_2$. If equations 32 have a solution at all $\mu_1 = \mu_2$; hence from section 6, the layered material is isotropic for long waves, and $L = M$, $RS = T^2$, and $(\frac{3}{4}L^{-1} - R)(\frac{3}{4}M - S) = (\frac{3}{4} - T)^2$. This case has already been discussed in section 6. In the remaining case we may assume $\mu_1 < \mu_2$, and $L < M$, $T^2 < RS$, $(\frac{3}{4} - T)^2 < (\frac{3}{4}L^{-1} - R)(\frac{3}{4}M - S)$. Momentarily taking μ_1 and μ_2 as known, we solve the first two of equations 32 for p_1 and p_2:

$$p_1 = \frac{\mu_2 - M}{\mu_2 - \mu_1} \qquad p_2 = \frac{M - \mu_1}{\mu_2 - \mu_1} \tag{33}$$

Then, from the third of equations 32,

$$\mu_1 + \mu_2 = M + L^{-1}\mu_1\mu_2 \tag{34}$$

We now solve the fourth and fifth of equations 32 for θ_1 and θ_2:

$$\theta_1 = \frac{T\mu_2 - S}{\mu_2 - M} \qquad \theta_2 = \frac{S - T\mu_1}{M - \mu_1} \tag{35}$$

Substituting (33) and (35) into the last of equations 32 we have

$$\mu_1 + \mu_2 = ST^{-1} + RT^{-1}\mu_1\mu_2 \tag{36}$$

Equations 34 and 36 can now be used to determine the rigidities μ_1 and μ_2. There are two possible cases. If $RL = T$, then (34) and (36) are soluble if and only if $S = MT$; by lemma 3, equations 32 also imply $L < M$, so that in this case we have the problem of constant θ, discussed in section 7.

In case $RL \neq T$, (34) and (36) can be solved for $\mu_1\mu_2$ and $\mu_1 + \mu_2$:

$$\mu_1 + \mu_2 = \frac{RLM - S}{RL - T} \qquad \mu_1\mu_2 = \frac{L(MT - S)}{RL - T}$$

It follows that μ_1 and μ_2 are the two roots of the quadratic equation

$$(RL - T)\mu^2 - (RLM - S)\mu + L(MT - S) = 0 \tag{37}$$

The only problems remaining in this second case are whether the roots μ_1 and μ_2 of (37) are real, positive, and different and when substituted into (33) and (35) whether the roots give $p_1 > 0$, $p_2 > 0$, $0 < \theta_1 < \frac{3}{4}$, and $0 < \theta_2 < \frac{3}{4}$. To discuss these questions we introduce a third set of elastic parameters, M, β, ξ, η, and ζ, defined thus:

$$\beta = LM^{-1} \qquad \xi = RM \qquad \eta = SM^{-1} \qquad \zeta = T$$

We let $r_1 = \mu_1/M$ and $r_2 = \mu_2/M$. Then (37) becomes

$$g(r) = 0 \tag{38}$$

where $g(r) = (\beta\xi - \zeta)\, r^2 - (\beta\xi - \eta)\, r + \beta(\zeta - \eta)$, equations 33 are

$$p_1 = \frac{r_2 - 1}{r_2 - r_1} \qquad p_2 = \frac{1 - r_1}{r_2 - r_1} \tag{39}$$

and equations 35 are

$$\theta_1 = \frac{\zeta r_2 - \eta}{r_2 - 1} \qquad \theta_2 = \frac{\eta - \zeta r_1}{1 - r_1} \tag{40}$$

The value of p_1, p_2, θ_1, θ_2, μ_1, and μ_2 are acceptable if and only if $0 < r_1 < 1 < r_2$, $r_1 < \eta\zeta^{-1} < r_2$, and $r_1 < (\frac{3}{4} - \eta)(\frac{3}{4} - \zeta)^{-1} < r_2$. Therefore, we must have either $g(0) > 0$, $g(1) < 0$, $g(\eta/\zeta) < 0$, $g(3 - 4\eta/3 - 4\zeta) < 0$, and $g(+\infty) < 0$, or $g(0) < 0$, $g(1) > 0$, $g(\eta/\zeta) > 0$, $g(3 - 4\eta/3 - 4\zeta) > 0$, and $g(+\infty) < 0$. The first set of conditions is

$$\beta(\zeta - \eta) > 0 \qquad (1 - \beta)(\zeta - \eta) > 0$$
$$\beta\xi - \zeta > 0 \qquad (\zeta - \eta)(\xi\eta - \zeta^2) > 0$$
$$\beta(\zeta - \eta)[(\tfrac{3}{4}\beta^{-1} - \xi)(\tfrac{3}{4} - \eta) - (\tfrac{3}{4} - \zeta)^2] > 0$$

The second set is the same as that given with $>$ replaced by $<$ throughout. A set of conditions equivalent to the first set is

$$0 < \beta < 1 \qquad \eta < \zeta < \beta\xi \qquad \xi\eta > \zeta^2$$
$$(\tfrac{3}{4}\beta^{-1} - \xi)(\tfrac{3}{4} - \eta) > (\tfrac{3}{4} - \zeta)^2 \tag{41}$$

A set of conditions equivalent to the second set is

$$0 < \beta < 1 \qquad \beta\xi < \zeta < \eta \qquad \xi\eta > \zeta^2$$
$$(\tfrac{3}{4}\beta^{-1} - \xi)(\tfrac{3}{4} - \eta) > (\tfrac{3}{4} - \zeta)^2 \tag{42}$$

Either the first set (41) or the second set (42), together with $M > 0$, is necessary and sufficient for the existence of a physically acceptable solution of equations 32 with $\mu_1 < \mu_2$ and $\theta_1 \neq \theta_2$; and if such a solution exists, it is unique.

To summarize the case just discussed, we have *Theorem 2*. A HTI medium with elastic parameters L, M, R, S, and T is long-wave equivalent to a strictly stable, two-layered, isotropic medium if and only if the parameters satisfy one of the following sets of conditions:

Set 1:

$$\left.\begin{array}{l} 0 < R < \tfrac{3}{4}L^{-1} \qquad 0 < S < \tfrac{3}{4}M \\ \qquad 0 < T < \tfrac{3}{4} \qquad T^2 < RS \\ (\tfrac{3}{4} - T)^2 < (\tfrac{3}{4}L^{-1} - R)(\tfrac{3}{4}M - S) \end{array}\right\} \tag{43}$$

$$SM^{-1} < T < RL \tag{44}$$

Set 2: The same as sét 1 except that (44) is replaced by

$$RL < T < SM^{-1} \tag{45}$$

Set 3:

$$0 < R < \tfrac{3}{4}L^{-1} \qquad 0 < S < \tfrac{3}{4}M$$
$$0 < T < \tfrac{3}{4} \qquad L < M$$
$$RL = T = SM^{-1} \tag{46}$$

Set 4:

$$0 < R < \tfrac{3}{4}L^{-1} \qquad 0 < S < \tfrac{3}{4}M$$
$$0 < T < \tfrac{3}{4} \qquad L = M$$
$$RL = T = SM^{-1}$$

In either set 1 or set 2, μ_1, μ_2, p_1, p_2, θ_1, and θ_2 are uniquely determined by (33), (35), (37), and the demand $\mu_1 < \mu_2$. In set 3, μ_1 and μ_2 are not uniquely determined, but are any solutions of (34) (whence $\mu_1 \neq \mu_2$), while p_1 and p_2 are uniquely determined by μ_1 and μ_2, and $\theta_1 = \theta_2 = T$. In set 4, $\mu_1 = \mu_2 = M$ and p_1, θ_1, p_2, and θ_2 can have any values consistent with $p_1 + p_2 = 1$ and $p_1\theta_1 + p_2\theta_2 = T$.

Conditions 43 in theorem 2 are the same as conditions 27 in theorem 1, while conditions 44, 45, or 46 impose further restrictions. We conclude that there are stable, three-layered, isotropic media which are not long-wave equivalent to any stable, isotropic, two-layered medium.

In consequence of theorems 1 and 2, it is clear that there is a hierarchy of stable, transversely isotropic media as follows: there are stable HTI media not long-wave equivalent to any stable, finely layered, isotropic medium. Every stable, finely layered, isotropic medium is long-wave equivalent to a stable, three-layered, isotropic medium. There are stable, three-layered, isotropic media not long-wave equivalent to any stable, two-layered, isotropic medium.

10. Comparison with Observation

Uhrig and Van Melle [1955] find that the horizontal velocity, $c_\parallel$, of compression waves is larger than the vertical velocity, $c_\perp$, by a factor of 1.17 to 1.40 in certain layers whose surface outcrops are homogeneous. They think this discrepancy probably represents an intrinsic anisotropy. In 1700 to 8000 feet of clastic and carbonate sediments, on the other hand, they find $c_\parallel/c_\perp = 1.10$ to 1.19. Now $c_\parallel^2 = A/\langle\rho\rangle$ and $c_\perp^2 = C/\langle\rho\rangle$ in a transversely isotropic medium with vertical axis of symmetry; therefore $(c_\parallel/c_\perp)^2 = A/C$. The question is whether observed variations of θ and μ with depth are able to account for values of A/C as large as 1.21 and 1.42.

The data on variation of θ and μ with depth come from well logs which measure the variation with depth of vertical compressional velocity, averaged over distances of 5 feet or more. For example, *Summers and Broding* [1952] find vertical compressional velocities in a single drill hole which vary more or less randomly from 9 to 15 kilofeet/sec. If density is assumed constant (they give no data on densities), and if it is assumed that the rock is homogeneous and isotropic over vertical distances of 5 feet or less, these variations in compressional velocity can be interpreted as variations in $\lambda + 2\mu = \mu\theta^{-1}$.

Can we explain values of $(c_\parallel/c_\perp)^2$ as large as Uhrig and Van Melle's on the basis of variations in $\mu\theta^{-1}$ as large as those obtained by Summers and Broding? (It would be much more satisfactory if both kinds of data were available for the same rock layer, but the present author knows of no such combined measurements and would greatly appreciate having any such called to his attention.) For simplicity, we restrict attention to a two-layered isotropic medium, in which both materials have the same density. We define $\alpha = \mu_2\theta_1/\mu_1\theta_2 > 1$, the ratio of the compressional velocities in the two media. On the basis of Summers and Broding's data we permit α to be as large as $(15/9)^2 = 2.78$. The expression for A/C in terms of the proportions p_1 and p_2 of the two materials and their elastic constants θ_1, θ_2, μ_1, and μ_2 is, from (32),

$$\frac{A}{C} - 1 = \frac{4p_1p_2}{\alpha}\left[\left(\frac{\alpha - 1}{2}\right)^2 - \left(\alpha\theta_2 - \theta_1 - \frac{\alpha - 1}{2}\right)^2\right] \quad (47)$$

Clearly, if α, θ_1, and θ_2 are fixed, A/C will be largest when $p_1 = p_2 = \frac{1}{2}$. If θ_1 and θ_2 are varied, the maximum value of A/C is

$$A/C = 1 + (\alpha - 1)^2/4\alpha \quad (48)$$

whereas if $\theta_1 = \theta_2 = \frac{1}{3}$ (Poisson's ratio $= \frac{1}{4}$),

$$A/C = 1 + 8(\alpha - 1)^2/36\alpha \quad (49)$$

From (48) and (49) it is clear that variations in θ will not markedly increase the anisotropy, $(A/C) - 1$. Furthermore, if A/C is to be appreciably larger than 1, the ratio α between the elastic constants of the two media must be very much larger than 1.

We assume that $\theta_1 = \theta_2 = \frac{1}{3}$. Then, with $\alpha = 2.78$, $A/C = 1.25$.

As far as the very rough calculations given above are concerned, we conclude that Uhrig and Van Melle's data on anisotropy in carbonate sediments can, with a bit of stretching, be explained by fine layering of an isotropic two-layered medium. More careful calculations, in which actual measured values of μ and θ are inserted into equations 18, are probably not justified until both well log measurements and gross anisotropy measurements are available for the same suite of rocks.

In making such comparisons of large-scale anisotropy with small-scale layering, it should be remembered that the layers do not become less effective in generating large-scale anisotropy as they become thinner. A wave 5 meters long in a medium made by laminating brass and steel 'sees' the same five effective anisotropic elastic constants, whether the lamina are 5 cm or 0.1 mm thick. Therefore, it is conceivable that layers too thin to be observed by contemporary well logging techniques are present and contribute appreciably to the large-scale anisotropy of the crust.

11. Summary of Conclusions

A transversely isotropic, stratified medium has been considered, whose axis of symmetry is the x_3 axis and whose properties vary only with x_3, not with x_1 or x_2. The medium may be locally isotropic. A length l' is chosen arbitrarily. (The results which follow are true for any l', but are useful only if l' is large enough so that the properties of the medium are significantly smoothed by averaging over a vertical distance l'.) The response of the medium to elastic waves whose wave numbers k are much less than $2\pi/l'$ can be calculated as follows: the medium is replaced by a 'long-wave equivalent' transversely isotropic medium, whose density is the average density (averaged locally over a vertical length l', and whose five elastic parameters are calculated from the parameters of the original medium by means of equations 9, the averages being computed locally over a vertical distance l'. The response of the resulting smoothed medium to waves of the given wave number k is calculated. The stresses and strains in this smoothed medium are the local average stresses and strains in the original medium, averaged over a vertical length l'. In the original medium the stresses T_{13}, T_{23}, T_{33} and the displacements s_1, s_2, s_3, and their derivatives with respect to x_1 and x_2 (but not x_3), are equal to their averages, while the stresses T_{11}, T_{22}, T_{12} and the displacement derivatives $\partial s_1/\partial x_3$, $\partial s_2/\partial x_3$, $\partial s_3/\partial x_3$ can be calculated from formulas 6. The above statements are only approximately correct. Where the error is known from exact treatments, it is of order $(kl')^2$, and in general it is probably of order kl'.

Detailed attention is given to the case of a finely layered medium which is locally isotropic. In this case, the five elastic parameters of the long-wave equivalent medium are given in terms of the Lamé parameters of the original medium by equations 13. (The author is indebted to D. Anderson for pointing out that equations 13 are not new, but were obtained by K. Helbig in 1958. The derivation given in the present paper is different from Helbig's and perhaps simpler.) The following question is then examined: Which stable, homogeneous, transversely isotropic media are long-wave equivalent to horizontally layered isotropic media? In other words, given the five elastic coefficients of a homogeneous, transversely isotropic medium, obtained from the response of the medium to long waves, when is it a possibility that shorter waves will reveal that the anisotropy is really the result of a fine layering of isotropic material?

To answer this question, new elastic parameters for transversely isotropic and for isotropic media are defined by equations 14 and 16. In terms of these new parameters, the relation between a layered isotropic medium and the long-wave equivalent, transversely isotropic medium is expressed by the very simple equations 18. The conditions for stability (a positive-definite internal energy) of a transversely isotropic medium are shown to be inequalities (21). Inequalities are then derived for the elastic coefficients of a transversely isotropic medium which are necessary and sufficient for the medium to be long-wave equivalent to a layered, isotropic, stable medium. These inequalities are (27). Since inequalities (27) are more restrictive than inequalities (21), it is concluded that there are stable, transversely isotropic media whose anisotropy cannot be the result of a layering of isotropic materials.

In the light of the above remarks, a natural question is this: Given a stable, homogeneous, transversely isotropic medium, how many different homogeneous isotropic materials are required to make a long-wave equivalent, layered, isotropic medium? In the present paper, we show that if the homogeneous, transversely isotropic medium is long-wave equivalent to any layered, isotropic medium, it is equivalent to a layered, isotropic medium made of just four homogeneous, isotropic materials. It can be shown that three suffice. Two do not suffice, and we derive conditions on the homogeneous, transversely isotropic material which are necessary and sufficient for it to be long-wave equivalent to a layered, isotropic medium made of just two homogeneous, isotropic materials. These conditions are (43), (44), (45) and (46).

In the course of the above arguments, it is shown that a layered, isotropic medium is long-wave equivalent to a homogeneous, isotropic medium if and only if it has constant rigidity (the 'if' half of this statement has been proved by *Postma* [1955] for two-layered media).

Finally, it is shown that some of the observed anisotropies in P velocity can be explained as due to layering of isotropic media, if contrasts are allowed to be as large as those observed in well logs.

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(Manuscript received May 31, 1962; revised July 19, 1962.)